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The critical level for hydromagnetic waves in a rotating fluid

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The propagation of plane hydromagnetic waves in a fluid rotating with angular velocity $\mathbf{\Omega}$ and permeated by a magnetic field $\mathbf{B} = \{B_x(z), B_y(z), 0\}$ varying in both magnitude and direction with z is studied by techniques recently applied to the propagation of internal gravity waves in a shear flow (Bretherton 1966; Booker & Bretherton 1967). Particular attention is paid to a class of 'slow' hydromagnetic waves of interest in connexion with the dynamics of the earth's liquid core. While, in general, rotation permits propagation *across* the lines of force, there is associated with each wave a 'critical level' $z = z_c$ which acts as a valve by effectively permitting the wave to penetrate it *from one side only*. A slow hydromagnetic wave with frequency ω and wavenumber components k, l normal to the magnetic field gradient can only effectively penetrate its critical level if its propagation speed across field lines W is such that $W\Omega_z(\Omega_x k + \Omega_y l)\omega < 0$. The phenomenon of 'critical-layer absorption' evidently does not in general require the presence of a mean shear flow; a non-uniform magnetic field gives rise to similar effects provided that some other restoring mechanism (in this case the Coriolis force) is available to permit hydromagnetic waves to propagate across field lines.

1. Introduction

When the effects of viscous and ohmic dissipation may be ignored the propagation of small amplitude plane hydromagnetic waves in an unbounded incompressible fluid rotating with constant angular velocity $\mathbf{\Omega}$ and permeated by a uniform magnetic field $\mathbf{B}$ is governed by the dispersion relationship

$$\omega^2 \pm \frac{2\mathbf{\Omega}.\boldsymbol{\kappa}}{\kappa}\omega - (\mathbf{V}.\boldsymbol{\kappa})^2 = 0 \tag{1.1}$$

(Lehnert 1954; Hide 1969). Here ω is the angular frequency of the waves, $\boldsymbol{\kappa} = (k, l, m)$ is the wavenumber vector, $\kappa = |\boldsymbol{\kappa}|$ and

$$\mathbf{V} = \mathbf{B}/(\mu\rho)^{\frac{1}{2}} \tag{1.2}$$

is the Alfvén velocity, where μ is the magnetic permeability and ρ the fluid density (assumed uniform).

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In the absence of rotation (1.1) yields Alfvén waves for which

$$\omega = \pm \mathbf{V}.\boldsymbol{\kappa} \tag{1.3}$$

(see Alfvén & Fälthammar 1963). A group of such waves propagates non-dispersively along the lines of force with velocity $\partial\omega/\partial\boldsymbol{\kappa} = \mathbf{V}$. According to (1.1) rotation permits hydromagnetic wave groups to propagate *across* field lines and renders them dispersive. The character of the waves depends on the magnitude of the parameter

$$Q = V\kappa/\Omega \tag{1.4}$$

and when $Q \ll 1$ the 'rapid' rotation has the effect of widely separating the roots of (1.1). One root is essentially an inertial wave (for which the restoring effect of the Coriolis force is dominant) which propagates very much faster than the Alfvén speed by a factor of order Q^{-1}:

$$\omega \doteqdot \pm 2\boldsymbol{\Omega}.\boldsymbol{\kappa}/\kappa \tag{1.5}$$

(see Greenspan 1968). The other is a hybrid 'hydromagnetic–inertial' wave (for which rotational and hydromagnetic restoring forces are equally important) which propagates very much *slower* than the Alfvén speed by a factor of order Q^{-1}:

$$\omega \doteqdot \pm (\mathbf{V}.\boldsymbol{\kappa})^2\kappa/(2\boldsymbol{\Omega}.\boldsymbol{\kappa}). \tag{1.6}$$

A certain amount of interest in hydromagnetic–inertial waves, and indeed in the whole comparatively undeveloped field of the magnetohydrodynamics of rotating fluids, has been stimulated recently by problems associated with the origin of the earth's magnetism. Hide (1966) concluded that the effects of the earth's rotation should be sufficiently strong for these slow hydromagnetic waves to propagate within the liquid core ($Q \sim 10^{-2}$) and that such waves should have periods commensurate with the (decades-to-centuries) time scale of the geomagnetic secular variation. Furthermore, by taking the effects of the spherical boundaries of the core into account by means of a simplified model, he suggested that those slow waves *characterized by quasi-two-dimensional motions* (in which fluid filaments parallel to the rotation axis move as coherent units) would propagate westward, and might accordingly contribute to the westward drift with time of the non-dipole geomagnetic field.

While the results of subsequent analyses (Stewartson 1967; Malkus 1967) do not appear to deny that waves with such '*filamentary*' motions will propagate westward in this way (Acheson 1971; Hide & Stewartson 1972) they make clear that three-dimensional modes of oscillation are possible and that these are not necessarily characterized by westward propagation. The present author has recently proposed a westward selection mechanism invoking hydromagnetic instabilities in a rotating fluid arising from spatial variations in the magnetic field (Acheson 1972*a*). (The magnetic field within the core of the earth is believed to be predominantly azimuthal, increasing from about 5 gauss in the neighbourhood of the core–mantle boundary to perhaps 100 gauss or so deeper within the core.) This paper is concerned with another effect due to spatial variations in the magnetic field, namely the restrictions such variations can impose on the propagation of hydromagnetic waves *across* field lines which, as we have already seen, takes place only by virtue of the rotation.

2. Equations of the problem

The basic equations of magnetohydrodynamics for a perfectly conducting, inviscid incompressible fluid of uniform density ρ referred to a co-ordinate system rotating with angular velocity $\mathbf{\Omega}$ are

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}.\nabla)\mathbf{u}+2\mathbf{\Omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p+\frac{1}{\mu\rho}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0, \tag{2.2}$$

$$\nabla.\mathbf{B} = 0, \tag{2.3}$$

$$\partial\mathbf{B}/\partial t = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{2.4}$$

where $\mathbf{u}$ denotes the Eulerian flow velocity relative to the rotating frame, p the pressure in excess of that required to balance the centrifugal force, $\mathbf{B}$ the magnetic field, μ magnetic permeability and t time (see Shercliff 1965; Hide 1969).

We wish to study wave propagation in a non-uniform magnetic field and it is mathematically expedient to permit the field to vary in one direction only. Thus on taking the z axis of a rectangular Cartesian co-ordinate system parallel to the magnetic field gradient (see figure 1) and seeking an equilibrium state of rigid-body rotation ($\mathbf{u} = 0$) about an axis *arbitrarily* inclined with that direction we arrive at two possibilities for the equilibrium magnetic field $\mathbf{B}_0(z)$ consistent with the equations of motion:

$$\mathbf{B}_0(z)\begin{cases} = \{B_x(z),\, B_y(z),\, 0\} & B_x,\, B_y \text{ arbitrary},\\ = \{B_x(z),\, B_y(z),\, B_z\} & B_x,\, B_y \text{ linear};\ B_z \text{ constant}.\end{cases}$$

Preferring to keep the magnetic field profile as general as possible, we elect to study perturbations about the first of these two equilibrium states. Note, however, that by so doing we exclude from the basic state a magnetic field component in the direction of the magnetic field gradient. The fluid has been assumed to be of constant density, so gravitational effects play no role in this problem and we shall for convenience speak of the magnetic field gradient as being 'vertical' and likewise refer to the basic magnetic field itself as 'horizontal'.

Consider now a small departure $\mathbf{u} = (u, v, w)$ from rigid-body rotation such that the inertia term in (2.1) may be neglected in comparison with the Coriolis term. This relative motion will be accompanied by a small perturbation

$$\mathbf{b} = (b_x, b_y, b_z)$$

to the magnetic field $\mathbf{B}_0(z)$. Equations (2.1)–(2.4) then reduce to a set of linear partial differential equations and admit plane-wave solutions in which all perturbation quantities ψ may be written as

$$\psi = \mathscr{R}[\hat{\psi}(z)\exp i(kx+ly-\omega t)]. \tag{2.5}$$

Elimination of all variables but $\hat{w}$ leads to the equation

$$(P^2-R^2)\hat{w}''+(R^2P'/P+PP'-2iRT)\hat{w}'+\{T^2-P^2(k^2+l^2)+iRP'T/P\}\hat{w} = 0, \tag{2.6}$$

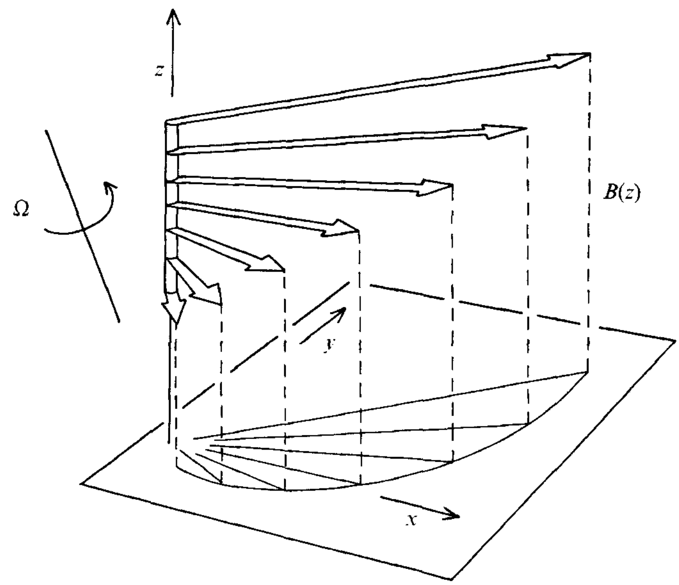

FIGURE 1. An observer rotating about the axis shown with angular velocity Ω will see a stationary fluid permeated by a stationary magnetic field varying in both magnitude and direction. A rectangular Cartesian co-ordinate system is chosen so that the z axis coincides with the (one) direction in which the magnetic field varies. There is no basic magnetic field component in this direction.

where
$$P(z) = (V_x k + V_y l)^2/\omega^2 - 1, \quad (2.7)$$
$$R = 2\Omega_z/\omega, \quad (2.8)$$
$$T = 2(\Omega_x k + \Omega_y l)/\omega, \quad (2.9)$$
$$V_x(z) = B_x(z)/(\mu\rho)^{\frac{1}{2}}, \quad V_y(z) = B_y(z)/(\mu\rho)^{\frac{1}{2}} \quad (2.10)$$

and primes denote differentiation with respect to z. Other perturbation quantities are related to $\hat{w}$ by means of the formulae
$$\hat{u} = [iTl\hat{w} + (Pik + Rl)\hat{w}']/P(k^2+l^2), \quad (2.11)$$
$$\hat{v} = [-iTk\hat{w} + (Pil - Rk)\hat{w}']/P(k^2+l^2), \quad (2.12)$$
$$\omega\hat{b}_x = -(B_x k + B_y l)\hat{u} - i\hat{w}B_x', \quad (2.13)$$
$$\omega\hat{b}_y = -(B_x k + B_y l)\hat{v} - i\hat{w}B_y', \quad (2.14)$$
$$\omega\hat{b}_z = -(B_x k + B_y l)\hat{w}. \quad (2.15)$$

When the magnetic field is uniform the coefficients of (2.6) are constant and the equation has solutions $\hat{w} \propto \exp(imz)$, where m is a constant vertical wavenumber satisfying (1.1).

Perhaps the most striking feature of (2.6) when the magnetic field is *not* uniform is its singularity at a height z_c where the field $\mathbf{B}_0(z)$ reaches a critical value such that $P^2 = R^2$, i.e.
$$[B_x(z_c)k + B_y(z_c)l]^2/\mu\rho\omega^2 - 1 = \pm 2\Omega_z/\omega. \quad (2.16)$$

The investigation of wave propagation across field lines in the neighbourhood of this 'critical level' is the main aim of the present paper.

3. The critical level

The singularity of (2.6) at the critical level may evidently be regarded as a consequence of the loss of higher derivatives owing to the neglect of dissipative effects, and can presumably be resolved by a boundary-layer-type analysis. Alternatively, as Miles (1961) has pointed out in connexion with the critical level for internal gravity waves in a shear flow (see Booker & Bretherton 1967), it may be regarded as a consequence of our restricted attention to a single sinusoidal component given by (2.5). Accordingly, by posing an initial-value problem and then determining its asymptotic solution as $t \to \infty$ we should be able to match the solutions on the two sides of the critical level (even when dissipative effects remain excluded). It has, however, proved possible to resolve the singularity by simpler means following Booker & Bretherton (1967). The method involves allowing the frequency ω to have a small imaginary part $\omega_i > 0$ so that the amplitude of the wave at any station is slowly growing with time. By thus investigating the solutions of (2.6) near $z = z_c$ and *then* taking the limit $\omega_i \to 0$ we obtain a matching condition connecting the solutions on the two sides of the critical level. The physical significance of this will be explained in due course, but we note here that the method is intimately related to the idea of a 'radiation condition' and may be thought of as modelling a slow, gradual switch-on of the source of the waves as opposed to the sudden switch-on common to many initial-value problems (Lighthill 1960, 1967).

If we write

$$\omega = \omega_r + i\omega_i \quad (\omega_i > 0), \tag{3.1}$$

then near the critical level, where the Alfvén velocity $\mathbf{V}(z)$ takes a special value such that

$$\frac{(V_x k + V_y l)^2_{z=z_c}}{\omega_r^2} - 1 = \pm \frac{2\Omega_z}{\omega_r}, \tag{3.2}$$

we find

$$\omega^2 P = \pm 2\Omega_z \omega_r + (z - z_c)\,[(V_x k + V_y l)^2]'_{z=z_c} - 2i\omega_r \omega_i + \omega_i^2, \tag{3.3}$$

where the upper or lower sign is taken according as the upper or lower sign is taken in (3.2). If ω_i is sufficiently small we may neglect the final term, but we retain the preceding term because it is imaginary. By multiplying (2.6) by ω^4 and inserting this expansion we find (on using (3.2)) that the coefficient of $\hat{w}''$ is

$$\pm 4\Omega_z \omega_r \{(z - z_c)\,[(V_x k + V_y l)^2]'_{z=z_c} - 2i\omega_i \{\Omega_z^2 + (V_x k + V_y l)^2_{z=z_c}\}^{\frac{1}{2}} \operatorname{sgn} \omega_r\}. \tag{3.4}$$

We therefore apply the method of Frobenius and seek a power-series solution such that near the critical level

$$\hat{w}(z) = C\left[z - z_c - \frac{2i\omega_i}{[(V_x k + V_y l)^2]'_{z=z_c}} \{\Omega_z^2 + (V_x k + V_y l)^2_{z=z_c}\}^{\frac{1}{2}} \operatorname{sgn} \omega_r\right]^{\chi}, \tag{3.5}$$

where C is an arbitrary constant. The indicial equation for χ has two roots and, if $\Omega_z \neq 0$, then

$$\chi = 0 \quad \text{or} \quad 2iA \operatorname{sgn}[P(z_c)R], \tag{3.6}$$

while if $\Omega_z = 0$
$$\chi = \pm 2iA, \tag{3.7}$$
where
$$A \equiv (\Omega_x k + \Omega_y l)\omega_r / [(V_x k + V_y l)^2]'_{z=z_c}. \tag{3.8}$$

Let us first concentrate on the second solution in (3.6), when $\Omega_z \neq 0$. As $z - z_c$ goes from positive values large compared with

$$L \equiv |2\omega_i / [(V_x k + V_y l)^2]'_{z=z_c}| \{\Omega_z^2 + (V_x k + V_y l)^2_{z=z_c}\}^{\frac{1}{2}}$$

through the critical level to values which are again large but negative, the argument of the complex quantity (3.4) changes smoothly through an angle of nearly π. If, having tracked the behaviour of $\hat{w}(z)$ smoothly through the critical level in this way, we finally let $\omega_i \to 0$ we obtain the matching condition that if

$$\left.\begin{aligned} &\hat{w}(z) = C \exp[2iA \operatorname{sgn}\{P(z_c)R\} \log(z - z_c)] \quad \text{for} \quad z > z_c \\ \text{then} \quad &\hat{w}(z) = C \exp[2iA \operatorname{sgn}\{P(z_c)R\} \log(z_c - z)] \\ &\qquad \times \exp[2|A|\pi \operatorname{sgn}\{\omega(\Omega_x k + \Omega_y l)\Omega_z P(z_c)\}] \quad \text{for} \quad z < z_c, \end{aligned}\right\} \tag{3.9}$$

so that the amplitudes on the two sides of the critical level differ by a factor of $\exp 2|A|\pi$.

We now seek to interpret this matching condition physically in terms of wave motion. Suppose for the sake of definiteness that $\omega(\Omega_x k + \Omega_y l)\Omega_z P(z_c) > 0$. Before taking the limit $\omega_i \to 0$ the amplitude of $\hat{w}(z)$ varies in a small neighbourhood of the critical level where $|z - z_c|$ is comparable with L, and in this neighbourhood the amplitude at any particular time is decreasing with height. Since $\omega_i > 0$, however, the amplitude at any given height is increasing with time and a given amplitude therefore propagates *upward* with time. In this sense the solution may be regarded as an upward travelling wave which is attenuated by a factor $\exp 2|A|\pi$ on passage through the critical level.

That this solution may be interpreted locally as an upward travelling wave may also be seen from energy considerations. The upward transfer of wave energy per unit area at any level will be the mean rate of working of the *total* pressure forces (including magnetic pressure) on the fluid above, i.e. $\overline{p_T w}$, where

$$p_T = p + \mu^{-1}(B_x b_x + B_y b_y) \tag{3.10}$$

and an overbar denotes an average over a horizontal wavelength. We may express $\hat{p}_T$ in terms of the velocity components alone:

$$\hat{p}_T = (\rho\omega/k)\{(2i/\omega)(\Omega_y \hat{w} - \Omega_z \hat{v}) - P\hat{u}\}, \tag{3.11}$$

and since
$$\overline{p_T w} = \tfrac{1}{4}(\hat{p}_T \tilde{w} + \tilde{p}_T \hat{w}), \tag{3.12}$$

where $\tilde{\psi}$ denotes the complex conjugate of $\hat{\psi}$, we may easily show (with the aid of (2.11) and (2.12)) that

$$\overline{p_T w} = \frac{\rho\omega}{4P(k^2 + l^2)}\{i(R^2 - P^2)(\hat{w}'\tilde{w} - \tilde{w}'\hat{w}) - 2RT\hat{w}\tilde{w}\}. \tag{3.13}$$

Thus on inserting the solution $\hat{w}(z) = \hat{w}_0 \exp[2iA \operatorname{sgn}\{P(z_c)R\} \log|z - z_c|]$ into (3.13) we find that

$$\overline{p_T w} = \frac{\rho|\hat{w}_0|^2}{k^2 + l^2}(\Omega_x k + \Omega_y l) \operatorname{sgn}\{P(z_c)\Omega_z \omega\}. \tag{3.14}$$

The amplitude $|\hat{w}_0|$ will, according to (3.9), be different on the two sides of the critical level, but in both cases energy flows *upward* with time.

If $\omega(\Omega_x k + \Omega_y l)\Omega_z P(z_c)$ is negative, both the above arguments lead to the conclusion that the second solution in (3.6) represents a *downward* travelling wave, again attenuated by a factor $\exp(2|A|\pi)$ on passage through the critical level. The energy flux in the neighbourhood of the critical level associated with the other solution in (3.6) ($\hat{w}(z)$ = constant, representing an unattenuated wave) is

$$\overline{p_T w} = -\frac{\rho|\hat{w}_0|^2}{k^2+l^2}(\Omega_x k + \Omega_y l)\,\mathrm{sgn}\,\{P(z_c)\Omega_z\omega\} \tag{3.15}$$

and is therefore opposite in direction to that associated with the solution we have just investigated (cf. equation (3.14)). When $\Omega_z \neq 0$ we therefore conclude that a wave crossing its critical level at a station $z = z_c$ such that

$$\{V_x(z_c)\,k + V_y(z_c)\,l\}^2/\omega^2 - 1 = \pm 2\Omega_z/\omega \tag{3.16}$$

will emerge either without attenuation or attenuated by a factor $\exp 2|A|\pi$, according as

$$W\Omega_z P(z_c)\,(\Omega_x k + \Omega_y l)\,\omega \lesseqgtr 0, \tag{3.17}$$

where W is its velocity of propagation in the z direction.

The author's main interest, for the reasons given in the introduction, lies in the slow 'hydromagnetic–inertial' waves characteristic of a 'rapidly' rotating fluid. Their frequency ω will in general be very small compared with Ω_z (see equation (1.6)) so that the critical level will be where

$$|V_x(z_c)\,k + V_y(z_c)l| \doteqdot |2\Omega_z\omega|^{\frac{1}{2}}. \tag{3.18}$$

We accordingly find that for these waves

$$|A| \doteqdot \left|\frac{\Omega_x k + \Omega_y l}{2\Omega_z}\frac{(V_x k + V_y l)^2_{z=z_c}}{[(V_x k + V_y l)^2]'_{z=z_c}}\right|. \tag{3.19}$$

Their attenuation factor will thus be very substantial provided only that the horizontal wavelengths involved are marginally less than the distance over which the magnetic field varies by a factor of order unity. They will therefore penetrate their critical level effectively *from one side only* (such that $W\Omega_z(\Omega_x k + \Omega_y l)\omega < 0$). This 'valve' effect is the main result of this paper.

If, on the other hand, the fluid is not rotating 'rapidly' (and ω/Ω_z is thus typically of order unity), there will in general be two distinct 'critical' values of $|V_x k + V_y l|$ given by (3.16). Since the sign of $P(z_c)\Omega_z\omega$ will be positive for one and negative for the other the two 'valves' will clearly work in opposition, one effectively transmitting only upward-moving waves and the other transmitting only downward-moving waves. Note that (for a given ω) the two critical levels become located closer to each other as Ω_z decreases. Indeed, as $\Omega_z \to 0$ it becomes increasingly difficult to resolve the individual 'valve' effects, for no sooner is a wave transmitted by one level than it is attenuated by the other. Finally, if $\Omega_z = 0$, the two critical levels merge into one at a level $z = z_c$ such that

$$\{V_x(z_c)\,k + V_y(z_c)l\}^2 = \omega^2. \tag{3.20}$$

Identical considerations to those above (which, we recall, have all been derived on the assumption that $\Omega_z \neq 0$) but based on the roots (3.7) of the indicial equation then lead to the conclusion that hydromagnetic waves are attenuated by a factor $\exp 2|A|\pi$ whichever side they come from. The 'valve' effect therefore depends essentially on the presence of a rotation component in the direction of the magnetic field gradient.

In the next section we look at these critical levels from a somewhat different point of view which is in some ways easier to picture physically. Before doing so, however, we remark that by differentiating (3.13) and using (2.6) and its complex conjugate as expressions for $\hat{w}''$ and $\tilde{w}''$ respectively it may be shown that

$$d(\overline{p_T w})/dz = 0, \tag{3.21}$$

so that *the mean flux of wave energy per unit area across field lines is independent of height.* This result holds for all magnetic field profiles and breaks down only at the critical level, where there is a discontinuity (see (3.9) and (3.14)). It has a simple interpretation when considered in the light of developments in the following section.

4. Wave propagation in a weak magnetic field gradient

When the magnetic field varies only slightly over distances of the order of a wavelength the concept of a *wave group* is extremely helpful. This is a time-dependent train of waves of sufficient regularity for a *local* frequency, wave-number and amplitude to be everywhere approximately defined, though these may vary with position and time. We shall focus attention on the propagation of these quantities rather than on the individual wave crests of which the train is composed. There is therefore a fundamental difference between this approach and that in the previous section, *where ω, k and l were fixed in space and time.* While it is natural that when the magnetic field varies only slightly over distances of the order of a wavelength the information contained in the *uniform*-field dispersion relation (1.1) should be of direct value, it is perhaps prudent first to clarify the *interpretation* of this relationship when the magnetic field varies with z.

If ω and $\boldsymbol{\kappa}$ vary with position and time (and $\mathbf{V}$ varies with z only) we may write (1.1) in the form

$$\omega = F(k, l, m, z) \tag{4.1}$$

and then formally define the 'group velocity'

$$\mathbf{u}_g = \left(\frac{\partial F}{\partial k}, \frac{\partial F}{\partial l}, \frac{\partial F}{\partial m}\right), \tag{4.2}$$

which therefore also varies with position and time. The significance of this group velocity as defined above (a result which we shall not prove here; see for example Whitham (1960), Lighthill (1965), Bretherton (1966)) is that an observer fixing his gaze on a point moving through the fluid everywhere with the local group velocity $\mathbf{u}_g$ will observe *constant values of ω, k and l.* Because the magnetic field varies with z, however, he will observe changes in the vertical wavenumber m. Thus the role of (1.1) when the magnetic field is not uniform is (a) to enable us, having selected the particular group we wish to track (i.e. having picked particular values of ω, k and l), to calculate the vertical wavenumber at each level in terms of the (known) magnetic field at that level and then (b) to permit us to use these values of m and $\mathbf{V}$ to calculate the group velocity at every level and in this way compute the trajectory of the group. We now use this procedure to study the propagation of wave groups in the neighbourhood of the critical level.

Writing the dispersion relation (1.1) as

$$\omega = \pm\frac{\Omega_x k+\Omega_y l+\Omega_z m}{(k^2+l^2+m^2)^{\frac{1}{2}}} \pm\left\{\frac{(\Omega_x k+\Omega_y l+\Omega_z m)^2}{k^2+l^2+m^2}+(V_x k+V_y l)^2\right\}^{\frac{1}{2}} \tag{4.3}$$

and then converting it to a formula for m in terms of the local Alfvén velocity $\mathbf{V}(z) = \{V_x(z), V_y(z), 0\}$, we find that

$$\begin{aligned}\{[(\mathbf{V}.\boldsymbol{\kappa})^2-\omega^2]^2-4\omega^2\Omega_z^2\}\,m^2-8\omega^2\Omega_z(\Omega_x k+\Omega_y l)\,m \\ +\{(k^2+l^2)\,[(\mathbf{V}.\boldsymbol{\kappa})^2-\omega^2]^2-4\omega^2(\Omega_x k+\Omega_y l)^2\} = 0.\end{aligned} \tag{4.4}$$

By confining attention first to the case $\Omega_z \neq 0$ it is clear that as the group approaches its critical level $z = z_c$, at which the Alfvén velocity takes a value $\mathbf{V}_c = \mathbf{V}(z_c)$ given by

$$\{(\mathbf{V}_c.\boldsymbol{\kappa})^2-\omega^2\}^2 = 4\omega^2\Omega_z^2, \tag{4.5}$$

one root of (4.4) increases indefinitely and is given asymptotically by

$$\{[(\mathbf{V}.\boldsymbol{\kappa})^2-\omega^2]^2-4\omega^2\Omega_z^2\}\,m = 8\omega^2\Omega_z(\Omega_x k+\Omega_y l). \tag{4.6}$$

The vertical component of group velocity w_g at any level can be computed from (4.3) and is given by

$$\begin{aligned}4\omega\kappa|\boldsymbol{\Omega}.\boldsymbol{\kappa}|\,\{(\mathbf{V}.\boldsymbol{\kappa})^2+(\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2\}^{\frac{1}{2}}\,w_g\,\mathrm{sgn}\,\{\omega^2-(\mathbf{V}.\boldsymbol{\kappa})^2\} \\ = \{4\omega^2\Omega_z^2-[(\mathbf{V}.\boldsymbol{\kappa})^2-\omega^2]^2\}m+4\omega^2\Omega_z(\Omega_x k+\Omega_y l).\end{aligned} \tag{4.7}$$

The asymptotic behaviour of w_g corresponding to the root (4.6) is therefore given by

$$4\omega m^2|\Omega_z|\,\{(\mathbf{V}_c.\boldsymbol{\kappa})^2+\Omega_z^2\}^{\frac{1}{2}}\,w_g\,\mathrm{sgn}\,\{\omega^2-(\mathbf{V}_c.\boldsymbol{\kappa})^2\} = -4\omega^2\Omega_z(\Omega_x k+\Omega_y l). \tag{4.8}$$

In the neighbourhood of the critical level

$$V_x(z) = V_x(z_c)+V_x'(z_c)\,(z-z_c)+\ldots,$$
$$V_y(z) = V_y(z_c)+V_y'(z_c)\,(z-z_c)+\ldots,$$

and from (4.6) we find that $|m|$ increases indefinitely as $|z-z_c|^{-1}$. According to (4.8) w_g then tends to zero as $(z-z_c)^2$ and this group is therefore neither transmitted nor reflected at its critical level but instead *never reaches it in a finite time.* As the critical level is approached

$$u_g \to \frac{(\mathbf{V}_c.\boldsymbol{\kappa})\,V_{xc}}{\{\Omega_z^2+(\mathbf{V}_c.\boldsymbol{\kappa})^2\}^{\frac{1}{2}}}\,\mathrm{sgn}\,\omega,\quad v_g \to \frac{(\mathbf{V}_c.\boldsymbol{\kappa})\,V_{yc}}{\{\Omega_z^2+(\mathbf{V}_c.\boldsymbol{\kappa})^2\}^{\frac{1}{2}}}\,\mathrm{sgn}\,\omega; \tag{4.9}$$

the group is therefore effectively 'captured' in this neighbourhood and constrained thereafter to propagate almost along the lines of force. This wave group clearly approaches the level from a direction such that

$$w_g\,\Omega_z\{(\mathbf{V}_c.\boldsymbol{\kappa})^2-\omega^2\}\,(\Omega_x k+\Omega_y l)\,\omega > 0. \tag{4.10}$$

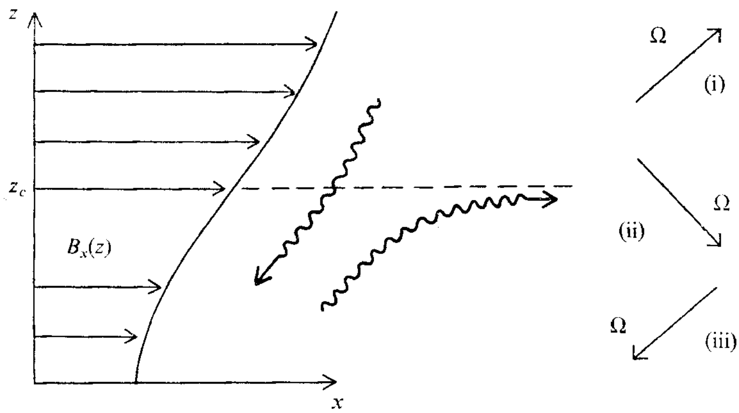

FIGURE 2. A simple example of a hydromagnetic–inertial wave group meeting a critical level. The system is supposed two-dimensional so that waves propagate in the x, z plane only. A wave group with frequency ω and horizontal wavenumber k will only penetrate its critical level if $w_g \Omega_z \Omega_x k\omega < 0$. If $\omega k > 0$ then in system (i) the wave can penetrate its critical level from above but not from below. This is the case illustrated. If $\omega k < 0$, on the other hand, the opposite is true. Even if ωk is positive the wave can still penetrate its critical level from below if the fluid rotates about a different axis (ii) (although the location of the critical level will then have changed), but merely reversing the sense of the rotation (iii) evidently does not alter which way the 'valve' works.

The other root of (4.4) tends to a finite value m_c as the critical level is approached (given by neglecting the m^2 term in (4.4)) and w_g then tends to a non-zero value given by

$$4\omega|\boldsymbol{\kappa}_c|\,|\boldsymbol{\Omega}.\boldsymbol{\kappa}_c|\,\{(\mathbf{V}_c.\boldsymbol{\kappa})^2+(\boldsymbol{\Omega}.\boldsymbol{\kappa}_c)^2/|\boldsymbol{\kappa}_c|^2\}^{\frac{1}{2}}\,\mathrm{sgn}\,[\omega^2-(\mathbf{V}_c.\boldsymbol{\kappa})^2]\,w_g = 4\omega^2\Omega_z(\Omega_x k+\Omega_y l), \tag{4.11}$$

where $\boldsymbol{\kappa}_c = (k, l, m_c)$. This wave, which approaches from a direction such that

$$w_g\,\Omega_z\{(\mathbf{V}_c.\boldsymbol{\kappa})^2-\omega^2\}\,(\Omega_x k+\Omega_y l)\,\omega < 0, \tag{4.12}$$

is therefore transmitted across the critical field line.

Thus, by focusing attention on a particular wave group by prescribing values of ω, k and l, we find that as it approaches its critical level it will be transmitted in *one direction only* across the critical field line and that the criterion for transmission or 'capture' is

$$w_g\,\Omega_z\{(\mathbf{V}_c.\boldsymbol{\kappa})^2-\omega^2\}\,(\Omega_x k+\Omega_y l)\,\omega \lessgtr 0 \tag{4.13}$$

(cf. equation (3.17)).

For a hydromagnetic–inertial wave group in a 'rapidly' rotating fluid the critical Alfvén speed is given to a good approximation by

$$(\mathbf{V}_c.\boldsymbol{\kappa})^4 = 4\omega^2\Omega_z^2, \tag{4.14}$$

and there is transmission or 'capture' according as

$$w_g\,\Omega_z(\Omega_x k+\Omega_y l)\,\omega \lessgtr 0. \tag{4.15}$$

Critical levels are likely to exert an important influence on these waves; consider a 'slow' wave group with vertical wavenumber m_0 at a station where the local

Alfvén velocity is $\mathbf{V}_0$. The local dispersion relation (1.6) then relates m_0 to $\mathbf{V}_0$:

$$|\omega| = \frac{(\mathbf{V}_0 \cdot \boldsymbol{\kappa})^2 (k^2+l^2+m_0^2)^{\frac{1}{2}}}{2|\Omega_x k + \Omega_y l + \Omega_z m_0|}, \tag{4.16}$$

and we therefore find that $\mathbf{V}_0$ is related to the critical Alfvén velocity $\mathbf{V}_c$ by

$$\frac{(\mathbf{V}_c \cdot \boldsymbol{\kappa})^2}{(\mathbf{V}_0 \cdot \boldsymbol{\kappa})^2} = \frac{|\Omega_z|(k^2+l^2+m_0^2)^{\frac{1}{2}}}{|\Omega_x k + \Omega_y l + \Omega_z m_0|}. \tag{4.17}$$

Taking, typically, k, l and m_0 of comparable magnitude we conclude that $|\mathbf{V}_c|/|\mathbf{V}_0|$ will be of order unity, except when the rotation axis is almost normal to the magnetic field gradient.

When $\Omega_z = 0$ and the rotation axis *is* normal to the magnetic field gradient the critical Alfvén velocity is given by $(\mathbf{V}_c \cdot \boldsymbol{\kappa})^2 = \omega^2$ and a reconsideration of the above analysis (in which we assumed $\Omega_z \neq 0$) leads to the conclusion that waves approaching this level are 'captured' whichever side they come from; no transmission is possible. 'Slow' waves in a 'rapidly' rotating fluid will still encounter such levels in this case, but they will clearly be located where the magnetic field either (*a*) takes an atypically low value (i.e. $|\mathbf{V}_c| \sim |\omega|/\kappa \sim Q|\mathbf{V}_0|$, see (1.6)) or (*b*) becomes almost perpendicular to the horizontal wavenumber vector $(k, l, 0)$.

We note that the conclusions of this section are entirely consistent with those of the previous one, for although some transmission was then always possible the attenuation factor $\exp 2|A|\pi$ increases indefinitely in the short wavelength limit, upon which the above analysis is based (see equation (3.19)). While the results of that section indicate that the critical level acts effectively as a valve even for $|A| = 1$ (for the attenuation factor $\exp 2\pi$ is still very substantial), it is in fact desirable that $|A|$ takes a rather higher value than this if the interpretation of the local solutions in that section as travelling waves is to be fully substantiated. To see this, note that the solutions (3.9) are both highly oscillatory near $z = z_c$ and that they both have a local vertical wavenumber

$$m = 2A \operatorname{sgn}\{P(z_c)R\}/(z-z_c). \tag{4.18}$$

The local dispersion relation (4.4) in the slowly varying field case may be used to show that the vertical wavenumber of a group about to be 'captured' is precisely (4.18) and the identification of (3.9) as travelling waves made in §3 is thus strongly supported by this comparison. One must note, however, that *this wavenumber is a function of z*, so that we may compute the fractional change in wavenumber over a vertical wavelength by first differentiating

$$\delta m = -\frac{2A \operatorname{sgn}\{P(z_c)R\}}{(z-z_c)^2}\delta z \tag{4.19}$$

and then putting $\delta z = 2\pi/|m|$, so that

$$|\delta m/m| = \pi/|A|. \tag{4.20}$$

The significance of $|A|$ not being too small is then clear, for only then will the fractional change in vertical wavenumber over a vertical wavelength be reasonably small and the concept of a local vertical wavenumber meaningful.

These considerations therefore neatly relate the two different methods of attack employed in this paper. The picture presented in this section, with wave-numbers varying only very slightly over distances of the order of a wavelength (the essence of the 'weak' magnetic field gradient assumption) and complete 'capture' rather than partial transmission, arises naturally out of the picture presented in §3 when $|A| \gg 1$ (formally, as $|A| \rightarrow \infty$). We finally relate the two approaches further by considering the propagation of wave energy.

When the magnetic field varies only slightly over distances of the order of a wavelength we may use (2.11)–(2.15) and (4.3) to express the mean wave energy per unit volume

$$\bar{E} = \tfrac{1}{2}\rho\overline{(u^2+v^2+w^2)} + (1/2\mu)\,\overline{(b_x^2+b_y^2+b_z^2)} \tag{4.21}$$

in terms of the local values of $\omega, k, l, m, \mathbf{V}(z)$ and the vertical velocity amplitude $|w|$:

$$\bar{E} = \tfrac{1}{2}\rho|w|^2\frac{k^2+l^2+m^2}{k^2+l^2}\left\{1+\frac{(V_xk+V_yl)^2}{\omega^2}\right\}. \tag{4.22}$$

By a somewhat lengthy analysis [which will be omitted here in the interests of saving space, for it is identical in form to that of Bretherton (1966), see Acheson (1971)] it may be shown that

$$\partial\bar{E}/\partial t + \nabla.(\mathbf{u}_g\bar{E}) = 0, \tag{4.23}$$

so that, for any volume element $\delta\tau$ whose boundaries move everywhere with the local group velocity, $\bar{E}\delta\tau$ is conserved. Wave energy is therefore conserved as the group moves. This result may be simply correlated with (3.21) for, if instead of a group we consider once again a single sinusoidal component with ω, k and l fixed in space and time, then $\mathbf{u}_g$ (as defined by (4.2)), like m, becomes a function of z only, so also does $\bar{E}$, and (4.23) reduces to $\bar{E}w_g =$ constant. When the magnetic field varies only slightly over distances of the order of a wavelength we may readily show by using (3.13), (4.7), (4.3) and (4.22) that

$$\overline{p_T w} = \bar{E}w_g. \tag{4.24}$$

It therefore follows that $\overline{p_T w}$ is constant. The result (3.21) derived *for a single sinusoidal component* (but without the short wavelength approximation made here) is thus intimately related to the fact that in the present system wave energy is propagated with the group velocity. (For a system in which this is *not* so see Bretherton (1966), Bretherton & Garrett (1968).) In view of this one can therefore envisage situations in which the overall effect of the 'valve' action of critical levels for hydromagnetic waves in a rotating fluid could be a continuous build-up of energy in a particular region, reminiscent of the well-known 'greenhouse' effect.

5. Discussion

Comparison of the critical-level properties discussed above with those for internal gravity waves in a shear flow (Bretherton 1966; Booker & Bretherton 1967; Jones 1967) reveals certain novel features. Internal gravity waves with frequency ω and horizontal (in the usual sense: normal to gravity $\mathbf{g} = (0, 0, g)$)

wavenumber components k, l propagating in a shear flow $(U_x(z), U_y(z), 0)$ are highly attenuated at a level at which the Doppler-shifted frequency $\omega - U_x k - U_y l$ vanishes, and this attenuation takes place no matter how the group approaches its critical level. Contrast this with the results above: a hydromagnetic wave in a rotating fluid approaching its critical level from one side will be highly attenuated, but it if approaches from the other side it will be transmitted without attenuation. The flux of energy across field lines associated with the attenuated wave was found to be different on the two sides of the critical level (see (3.9) and (3.14)) and accordingly most of the energy of the wave must be absorbed there into, presumably, a change in the mean magnetic field in that neighbourhood, the establishment of a small mean flow or both. A full investigation of exactly what *does* become of the attenuated wave's energy has, however, not yet been carried out, although the author hopes to include such considerations in a future paper at present in preparation (Acheson 1972*b*).

Note that the phenomenon of critical-layer absorption does not appear to depend crucially on the presence of a mean shear flow. A non-uniform *magnetic field* is evidently capable of giving rise to critical-layer absorption provided that some other restoring mechanism is available to allow hydromagnetic wave energy to propagate across field lines. While this has been provided above by the Coriolis force due to the fluid's rotation, we shall see in a companion paper (Acheson 1972*b*) how the action of gravity on a *stratified* fluid can, through the action of the *buoyancy* force, accomplish similar effects.

To conclude that these critical levels invariably accompany hydromagnetic wave propagation in a rotating fluid would be premature. It is first necessary to show that their properties (and indeed their very *existence*) are not especially sensitive to small deviations from the strict assumptions of the present theory. The main assumptions are (*a*) that the fluid is inviscid and perfectly conducting (*b*) that the fluid is homogeneous and (*c*) that the magnetic field gradient is normal to the field itself.

Perhaps the most obvious inherent assumption is (*a*). In the short wavelength approximation of §4 the vertical wavenumber m increases indefinitely as a group is 'captured' so that in practice, rather than having the picture of a group taking an infinite time to reach the critical level but retaining its identity indefinitely, we have the picture of a group whose associated velocity shear rapidly increases until the effects of small but finite viscosity and electrical resistivity come into play and destroy the group as a coherent entity. Provided dissipative effects are in some sense small, however, the author does not expect the overall critical-level properties to be changed, as a numerical boundary-layer-type analysis for the critical layer for internal gravity waves leads to the *same* attenuation factor as that found by an analysis similar to that presented in §3 (Hazel 1967).

Relaxation of assumption (*b*) is vital if we wish to discuss these critical levels in the context of geophysical or astrophysical problems since the effects of density stratification are then often important. This will be one of the topics dealt with in the companion paper mentioned previously.

If the basic magnetic field has a constant vertical component B_z (which, as we have seen, is permissible if B_x and B_y are linear) then within the short wavelength approximation the local dispersion relation will take the form

$$\omega = \pm \frac{(\Omega_x k + \Omega_y l + \Omega_z m)}{(k^2+l^2+m^2)^{\frac{1}{2}}} \pm \left\{\frac{(\Omega_x k + \Omega_y l + \Omega_z m)^2}{k^2+l^2+m^2} + (V_x k + V_y l + V_z m)^2\right\}^{\frac{1}{2}}. \quad (5.1)$$

By focusing attention on a particular group (i.e. selecting ω, k and l) it becomes clear that however small B_z may be m can no longer increase indefinitely in any circumstances! This does not, of course, automatically imply that these critical-level phenomena are exceptional, for the WKB (short wavelength) approximation (by its asymptotic nature) is incapable of predicting partial transmission; it can predict only *total* transmission, reflexion or 'capture' (Bretherton 1966). In practice one expects that if B_z is in some sense small it will enable some transmission (just as a field $(B_x(z), B_y(z), 0)$ enables some transmission except in the short wavelength limit) but that the overall 'valve' effect will persist. We remark, however, in conclusion that *how* small the basic magnetic field component in the direction of the magnetic field gradient must be for the overall critical-level properties established above to be qualitatively correct is not known. The resolution of this problem should provide a key factor in the determination of how likely these critical levels are to occur in natural systems (such as the earth's liquid core) where the magnetic field configuration may be extremely complicated.

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REFERENCES

ACHESON, D. J. 1971 The magnetohydrodynamics of rotating fluids. Ph.D. thesis, University of East Anglia.

ACHESON, D. J. 1972*a* On the hydromagnetic stability of a rotating fluid annulus. *J. Fluid Mech.* **52**, 529.

ACHESON, D. J. 1972*b* Hydromagnetic wave propagation in a rotating stratified fluid. (in preparation).

ALFVÉN, H. & FÄLTHAMMAR, C.-G. 1963 *Cosmical Electrodynamics*, 2nd edn. Oxford University Press.

BOOKER, J. R. & BRETHERTON, F. P. 1967 The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* **27**, 513.

BRETHERTON, F. P. 1966 The propagation of groups of internal gravity waves in a shear flow. *Quart. J. Roy. Met. Soc.* **92**, 466.

BRETHERTON, F. P. & GARRETT, C. J. R. 1968 Wave trains in inhomogeneous moving media. *Proc. Roy. Soc.* A **302**, 529.

GREENSPAN, H. P. 1968 *The Theory of Rotating Fluids*. Cambridge University Press.

HAZEL, P. 1967 The effect of viscosity and heat conduction on internal gravity waves at a critical level. *J. Fluid Mech.* **30**, 775.

HIDE, R. 1966 Free hydromagnetic oscillations of the earth's core and the theory of the geomagnetic secular variation. *Phil. Trans. Roy. Soc.* A **259**, 615.

HIDE, R. 1969 On hydromagnetic waves in a stratified rotating incompressible fluid. *J. Fluid Mech.* **39**, 283.

HIDE, R. & STEWARTSON, K. 1972 Hydromagnetic oscillations of the earth's core. *Rev. Geophys. & Space Phys.* **10** (to appear).

JONES, W. L. 1967 Propagation of internal gravity waves in fluids with shear flow and rotation. *J. Fluid Mech.* **30**, 439.

LEHNERT, B. 1954 Magnetohydrodynamic waves under the action of the Coriolis force I. *Astrophys. J.* **119**, 647.

LIGHTHILL, M. J. 1960 Studies on magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans. Roy. Soc.* A **252**, 397.

LIGHTHILL, M. J. 1965 Group velocity. *J. Inst. Math. Appl.* **1**, 1.

LIGHTHILL, M. J. 1967 On waves generated in dispersive systems by travelling forcing effects, with application to the dynamics of rotating fluids. *J. Fluid Mech.* **27**, 725.

MALKUS, W. V. R. 1967 Hydromagnetic planetary waves. *J. Fluid Mech.* **28**, 793.

MILES, J. W. 1961 On the stability of heterogeneous shear flows. *J. Fluid Mech.* **10**, 496.

SHERCLIFF, J. A. 1965 *A Textbook of Magnetohydrodynamics*. Pergamon Press.

STEWARTSON, K. 1967 Slow oscillations of fluid in a rotating cavity in the presence of a toroidal magnetic field. *Proc. Roy. Soc.* A **299**, 173.

WHITHAM, G. B. 1960 A note on group velocity. *J. Fluid Mech.* **9**, 347.